Tollmien-Schlicting Wave Propagation

Tollmien-Schlichting (TS) waves arise in laminar-turbulent transition of boundary layer flows. In the initial stage of transition, small disturbances transform a stable laminar flow into an unstable, but still laminar, flow which takes the form of a two-dimensional Tollmien-Schlichting wave. In this example we solve a Tollmien-Schlichting wave problem with a base plane Poiseuille flow to examine the accuracy and convergence properties of AcuSolve.

The flow domain here is a rectangular channel with a half width of 1 and an axial length of π. The base flow has a laminar parabolic profile with a maximum velocity of 1. The dynamic viscosity is 1.333e-4, so that the half-channel width *Re* ~= 7500. The Tollmien-Schlichting wave is a traveling wave of small disturbances ($\hat{u}$, $\hat{v}$) on the base flow $U = 1 - y^2$. These fluctuations are given by

$$\hat{u} = \epsilon\phi'(y)e^{i(\alpha x - \beta t)}$$
$$\hat{v} = -i\epsilon\alpha\phi(y)e^{i(\alpha x - \beta t)}$$

We use ϵ=.001 here. ϕ and β are the solution of the eigenvalue problem given by the Orr-Sommerfeld equation

$$(U-c)(\phi'' - \alpha\phi) - U''\phi = -\frac{i}{\alpha Re}\left(\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi\right)$$

for homogeneous Dirichlet and Neumann boundary conditions arising from the no slip condition at the wall, where $c = \beta/\alpha$. The wavelength of the TS wave is 2 π α, and α is set to 1 so that the flow is periodic in the axial direction for our domain. For *Re*=7500 and α=1, the imaginary part of β is .002235, so we expect to see the magnitude of the disturbance ($\hat{u}$, $\hat{v}$) grow at that rate.

The Tollmien-Schlichting wave solution itself is a solution of the linearized Navier-Stokes equations, not the full nonlinear system solved by AcuSolve. For a given resolution, then, we use the computed growth rate to measure the accuracy of our simulation rather than the TS wave solution itself.

We compute the flow with AcuSolve on a rectangular grid with uniform spacing in the flow direction and cosine stretched spacing in the wall normal direction. The initial conditions of the simulation are the base flow plus the fluctuations at *t*=0. Three grid resolutions are used:

- 4h grid: 33x33, dt=4.19e-2
- 2h grid: 65x65, dt=2.09e-2
- h grid: 129x129, dt=1.05e-2

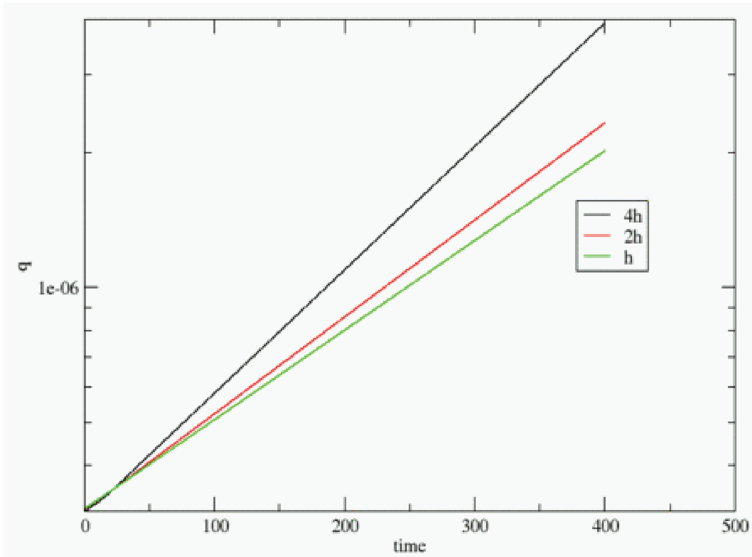

Figure 1

Figure 1 above shows the time evolution of the growth of the fluctuation for the three simulations. The animations below show the evolution of the transverse velocity perturbation (bottom animation) and the axial velocity perturbation (top) over a single period for the 2h mesh. Look carefully to see that the perturbations do indeed grow over the course of the single period. Note that since ϵ=.001, these perturbations at early time are roughly 0.1% of the mean flow in magnitude.

Table 1 summarizes the mesh averaged growth rate of the fluctuation and error for the 3 simulations. We fit the error to a power law Ch^p by comparing errors at successive resolutions; the values of *p* are also shown in Table 1.

Table 1. Results of convergence study

Grid resolution	Predicted growth rate	Error	% Error	Order (*p*)
4h	0.003175	0.000940	42.1%	-
2h	0.002495	0.000260	11.6%	1.85
h	0.002304	0.000069	3.1%	1.88

The table shows that AcuSolve is achieving second order accuracy for this flow, and also converging to the correct solution as the mesh is refined.