

Isentropic and Ideal Gas Density Relationships

Consider the subsonic flow of a real gas through a smooth duct with a contraction and expansion forming a gradually converging and diverging nozzle. An idealization of this type of flow is to assume that the flow is isentropic. i.e. entropy remains constant throughout, and that the gas is an ideal gas. It is a common practice to assume isentropic flow when friction effects are small, there is negligible heat transfer and there are no shocks. Since the flow is reversible and adiabatic, both the stagnation pressure and stagnation temperature are constant.

For an ideal gas, an isentropic flow obeys the equation

$$\frac{\rho}{\rho_t} = \left(\frac{P}{P_t} \right)^\gamma$$

where γ is the ratio of C_p , the specific heat at constant pressure, to C_v , the specific heat at constant volume; P is the pressure; and ρ is the density. The subscript t stands for the total (or stagnation) conditions. In the following example, *AcuSolve*TM is used to solve for the isentropic flow of air in a converging-diverging nozzle. The geometry is shown in Figure 1. The nozzle has constant and equal diameter inlet and outlet sections, commonly referred to as *reservoir* and *receiver* sections. Since we are neglecting viscous effects, the solution is obtained by setting the dynamic viscosity to a negligible value (practically zero) and slip boundary condition at the wall. The velocity at the inlet is 50 m/sec and pressure at the outlet is prescribed as atmospheric ($P_{atm} = 101325$ Pa). For the geometry chosen, the peak Mach number reached in the throat is $M \approx 0.63$.

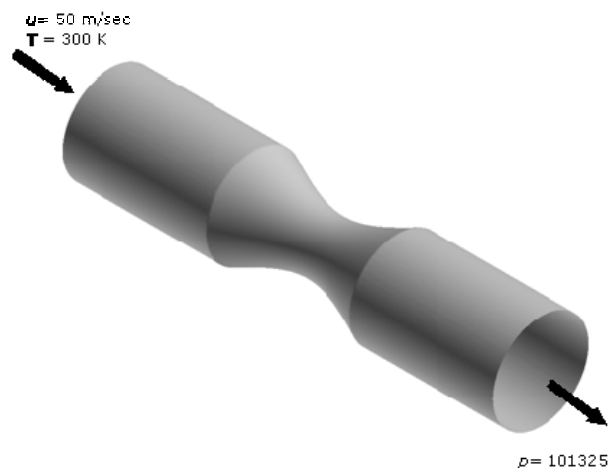


Figure 1. Nozzle geometry

Two different cases are solved based on the treatment of the equation of state for air.

Case 1. Isentropic flow is "assumed". Hence, the pressure and density are strict functions of each other.

$$\frac{P}{\rho^\gamma} = \text{constant}$$

The energy equation is not solved in this case.

Case 2. The ideal gas equation of state is used along with the energy equation, hence density is taken as a function of pressure and temperature using the following equation

$$\frac{P}{\rho} = RT$$

where $R = C_p - C_v$ is the gas constant and T is the temperature.

All calculations were performed using a three dimensional tetrahedral mesh with 134492 elements and 26274 nodes. Case 1 required 17 time steps for convergence while Case 2 required 35 time steps to reach convergence. In both cases, convergence is defined to be reached when the normalized residuals for all variables are $1e^{-5}$ or less. Due to the energy conservation equation the total number of time steps to reach convergence was higher for Case 2. Figure 2 shows the convergence histories for the mass and momentum residuals for both Case 1 and 2. The solution histories for Case 2 are similar for mass and momentum residuals.

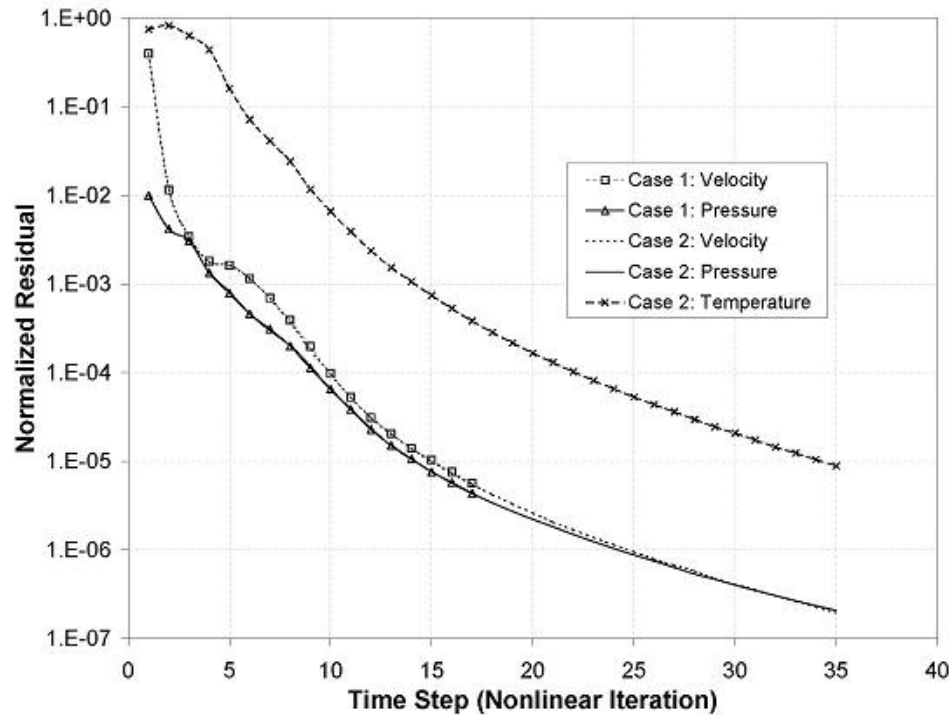


Figure 2. Normalized residuals vs. time step

The conservation of mass principle tells us that the mass flow rate (\dot{m}) through a tube remains constant. Also, for the solutions obtained here, the velocity profile across the nozzle is nearly uniform so the predicted flow is nearly one-dimensional. The mass flux (G) or mass flow rate per unit area varies inversely with the cross-sectional area. For a given \dot{m} we can calculate the average G from the local cross-sectional area (A_c).

$$G = \frac{\dot{m}}{A_c}$$

Figure 3 plots the average mass flux over the cross-section for the solutions of Case 1 and Case 2 as a function of axial position along the nozzle. For comparison, the mass flux calculated using the classical relationship for one-dimensional isentropic flow in a duct is also plotted [1].

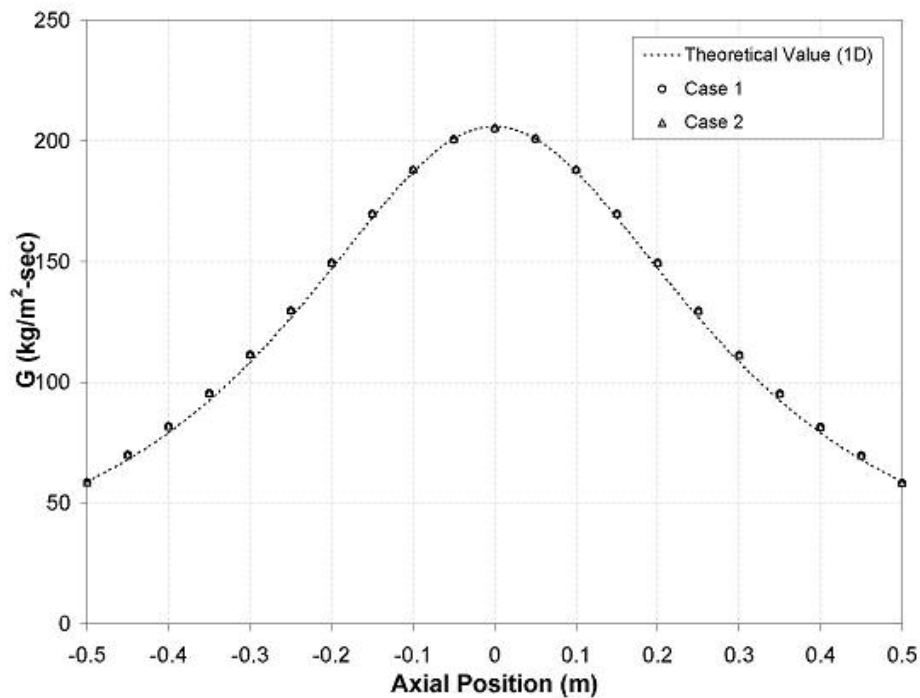


Figure 3. Average mass flux vs axial position

The Mach number (M) is the ratio of speed of the flow (u) to the speed of sound (c).

$$M = \frac{u}{c}$$

The speed of sound depends on the density (ρ), the pressure (P), the temperature (T) and the ratio of specific heats (γ)

$$c = \sqrt{\gamma \frac{P}{\rho}} = \sqrt{\gamma RT}$$

Hence, we can write

$$M = u \sqrt{\frac{\rho}{\gamma P}} = \frac{u}{\sqrt{\gamma RT}}$$

And the theoretical Mach number can be calculated from the mass flux, using the relationship

$$G = \sqrt{\gamma \rho_t P_t} M \left[1 + \frac{\gamma - 1}{2} M^2 \right]^{-\frac{\gamma + 1}{2(\gamma - 1)}}$$

The Mach numbers for both the numerical solutions along with the one-dimensional flow solution are plotted in Figure 4 as a function of axial position along the nozzle. The *AcuSolve*TM solutions are again quite similar to each other and to the classical flow solution.

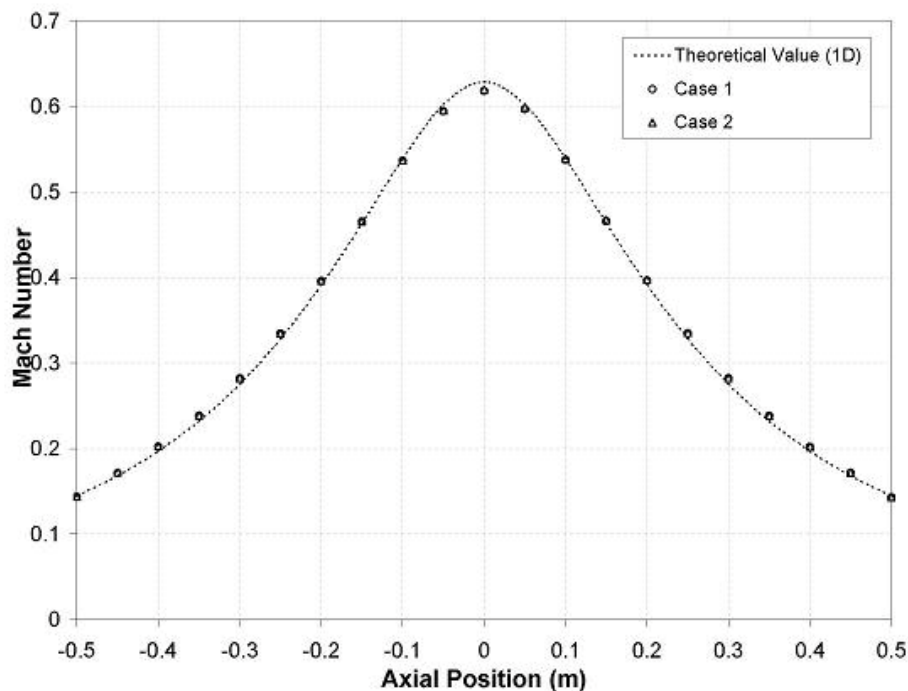


Figure 4. Average Mach number vs axial position

This exercise shows that good agreement is achieved between the two solutions of isentropic flow in a duct using *AcuSolve*TM, one using the isentropic density relationship and a second solving the energy equation and using ideal gas density relationship. Both of these solutions compare well with the classical one-dimensional solution for compressible flow in a duct.

This suggests that *AcuSolve*TM can be used to provide accurate solutions for density variation using isentropic and ideal gas relationships.

[1] See, for example, Asher H. Shapiro, Compressible Fluid Flow, Ronald Press, NY, 1953.